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Magnetized Cosmological Model

Raj Bali[1]

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The object of this paper is to investigate the behavior of the magnetic field in a cosmological model for perfect fluid distribution. The magnetic field is due to an electric current produced along the x axis. It is assumed that expansion (θ) in the model is proportional to σ_1^1, the eigenvalue of the shear tensor σ_i^j. The behavior of the model when the magnetic field tends to zero and other physical properties are also discussed.

1. INTRODUCTION

Bianchi type I cosmological models that are anisotropic homogeneous universes play an important role in understanding essential features of the universe, such as formation of galaxies during its early stages of evolution. An LRS Bianchi type I cosmological model containing a magnetic field directed along one axis with a barotropic fluid was investigated by Thorne (1967). Jacobs (1968, 1969) investigated Bianchi type I cosmological models with magnetic field satisfying a barotropic equation of state. Collins (1972) gave a qualitative analysis of Bianchi type I models with magnetic field. Roy and Prakash (1978) obtained a plane symmetric cosmological model with an incident magnetic field for perfect fluid distribution. In this paper I consider the magnetic flux vector h_i to be along the x direction. I further assume that σ_1^1, the eigenvalue of shear tensor σ_i^j in that direction, bears a constant ratio to the scalar expansion (θ). The distribution consists of an electrically neutral perfect fluid with an infinite electrical conductivity in the presence of a magnetic field.

Let us consider the space-time to be Bianchi type I, since this represents the simplest anisotropic homogeneous universe. The metric is taken in the form given by

$$ds^2 = A^2(dx^2 - dt^2) + B^2\, dy^2 + C^2\, dz^2 \tag{1}$$

[1]Department of Mathematics, University of Rajasthan, Jaipur 302004, India.

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where the metric potentials are functions of time alone. The energy-momentum tensor is taken into the form

$$T_i^j = (\varepsilon + p)v_i v^j + p g_i^j + E_i^j \tag{2}$$

where E_{ij} is the electromagnetic field given by Lichnerowicz (1967):

$$E_{ij} = \bar{u}[|h|^2(v_i v_j + \tfrac{1}{2}g_{ij}) - h_i h_j] \tag{3}$$

In the above ε is the density, p the pressure, and v_i the flow vector satisfying

$$g_{ij}v^i v^j = -1 \tag{4}$$

$\bar{\mu}$ is the magnetic permeability and h_i is the magnetic flux vector defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}}\varepsilon_{ijkl}F^{kl}v^j \tag{5a}$$

where F_{kl} is the electromagnetic field tensor and ε_{ijkl} the Levi-Civita tensor density. A semicolon stands for covariant differentiation. Let us assume the coordinates to be comoving, so that $v^1 = 0 = v^2 = v^3$ and $v^4 = 1/A$. Take the incident magnetic field to be in the direction of x axis, so that $h_1 \neq 0$, $h_2 = 0 = h_3 = h_4$. This leads to $F_{12} = F_{13} = 0$ by virtue of (5a). Also, $F_{14} = F_{24} = F_{34} = 0$ due to the assumption of the infinite conductivity of the fluid. Hence the only nonvanishing component of F_{ij} is F_{23}. The first set of Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0$$

leads to $F_{23} = \text{const} = H$ (say). Hence

$$h_1 = AH/\bar{\mu}BC \tag{5b}$$

The field equations

$$R_i^j - \tfrac{1}{2}Rg_i^j + \Lambda g_i^j = -8\pi T_i^j \tag{6}$$

for the line element (1) are

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4C_4}{BC} + \frac{A_4B_4}{AB} + \frac{A_4C_4}{AC}\right) - \Lambda = 8\pi\left(p - \frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{7}$$

$$\frac{1}{A^2}\left(-\frac{C_{44}}{C} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right) - \Lambda = 8\pi\left(p + \frac{H^2}{2\bar{u}B^2C^2}\right) \tag{8}$$

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right) - \Lambda = 8\pi\left(p + \frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{9}$$

$$\frac{1}{A^2}\left(\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC}\right) + \Lambda = 8\pi\left(\varepsilon + \frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{10}$$

2. SOLUTION OF THE FIELD EQUATION

Equations (7)-(10) are four equations in five unknowns, A, B, C, ε, and p. For the complete determination of this set, let us assume that expansion (θ) in the model is proportional to the eigenvalue σ_1^1 of the shear tensor σ_i^j. This condition leads to

$$A=(BC)^n \tag{11}$$

where n is constant. In this paper, we shall choose $n=1$, so that

$$A=BC \tag{12}$$

From equations (7)-(9), one has

$$\left(\frac{A_4}{A}\right)_4+\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)-\frac{B_{44}}{B}-\frac{B_4C_4}{BC}=-\frac{8\pi H^2A^2}{\bar{\mu}B^2C^2} \tag{13}$$

and

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}=0. \tag{14}$$

Using the condition (12) in equation (13) gives the following equation:

$$\frac{C_{44}}{C}+\frac{B_4C_4}{BC}=-\frac{8\pi H^2}{\bar{\mu}} \tag{15}$$

Setting $BC=\mu$ and $B/C=\nu$ in equations (14) and (15) gives

$$\mu_{44}-\left(\frac{\mu\nu_4}{\nu}\right)_4+\frac{16\pi H^2}{\bar{\mu}}=0 \tag{16}$$

and

$$\left(\frac{\mu\nu_4}{\nu}\right)_4=0 \tag{17}$$

Using equation (17) in equation (16) gives

$$\mu_{44}+a^2\mu=0 \tag{18}$$

where

$$a^2=16\pi H^2/\bar{\mu}$$

Equation (18) on integration leads to

$$\mu=\frac{b}{a}\sin\left[a(t+l)\right] \tag{19}$$

b^2 and l are constants of integration. Equations (17) and (19) lead to

$$\nu = \left(\frac{2}{a}\right)^{m/b}\left[\tan\frac{a(t+l)}{2}\right]^{m/b} \tag{20}$$

where m is a constant of integration and $(2/a)^{m/b}$ is an arbitrary constant. Hence the metric (1) reduces to the form

$$\begin{aligned} dS^2 = {} & \frac{b^2 \sin^2[a[t+l)]}{a^2}(dx^2 - dt^2) \\ & + \frac{b}{a}\sin[a(t+l)]\left(\frac{2}{a}\right)^{m/b}\left[\tan\frac{a(t+l)}{2}\right]^{m/b} dy^2 \\ & + \frac{b}{a}\sin[a(t+l)]\left(\frac{2}{a}\right)^{-m/b}\left[\tan\frac{a(t+l)}{2}\right]^{-m/b} dz^2 \end{aligned} \tag{21}$$

After suitable transformation of coordinates, the metric (21) reduces to the form

$$\begin{aligned} dS^2 = {} & \frac{\sin^2 aT}{a^2}(dX^2 - dT^2 + \left(\frac{2}{a}\right)^{\beta}\frac{\sin aT}{a}\left(\tan\frac{aT}{2}\right)^{\beta} dY^2 \\ & + \left(\frac{2}{a}\right)^{-\beta}\frac{\sin aT}{a}\left(\tan\frac{aT}{2}\right)^{-\beta} dZ^2 \end{aligned} \tag{22}$$

When the magnetic field, i.e., when $a \to 0$, then

$$dS^2 = T^2(dX^2 - dT^2) + T^{1+\beta}\,dY^2 + T^{1-\beta}\,dZ^2 \tag{23}$$

β is an arbitrary constant.

3. SOME PHYSICAL AND GEOMETRICAL FEATURES

The pressure and density for the model (22) are given by

$$8\pi p = \frac{a^2}{\sin^2 aT}\left[a^2\left(\frac{\beta^2}{4} + \tfrac{1}{2}\right)\operatorname{cosec}^2 aT - \frac{a^2}{4}\cot^2 aT\right] - \frac{a^4}{4\sin^2 aT} - \Lambda \tag{24}$$

and

$$8\pi\varepsilon = \frac{a^2}{\sin^2 aT}\left(\frac{5a^2}{4}\cot^2 aT - \frac{\beta^2 a^2}{4}\operatorname{cosec}^2 aT\right) - \frac{a^4}{4\sin^2 aT} + \Lambda \tag{25}$$

The model has to satisfy the following reality conditions:

1. $p > 0$.
2. $\varepsilon - p \geq 0$

Condition 1 leads to

$$a^4(\tfrac{1}{4}\beta^2 + \tfrac{1}{4})\operatorname{cosec}^2 aT - \Lambda \sin^2 aT > 0 \tag{26}$$

while condition 2 leads to

$$\operatorname{cosec}^4 aT - \frac{3}{2-\beta^2}\operatorname{cosec}^2 aT + \frac{4\Lambda}{a^4(2-\beta^2)} \geq 0 \tag{27}$$

From (27), we have

$$\left[\operatorname{cosec}^2 aT - \frac{3}{2(2-\beta^2)}\right]^2 - \left[\frac{9}{4(2-\beta^2)^2} - \frac{4\Lambda}{a^4(2-\beta^2)}\right] \geq 0 \tag{27a}$$

Here two cases arise:

Case 1: If $\Lambda \leq 0$, then condition (26) is automatically satisfied and

$$\operatorname{cosec}^2 aT > \frac{3}{2(2-\beta^2)} - \left[\frac{9}{4(2-\beta^2)^2} - \frac{4\Lambda}{a^4(2-\beta^2)}\right]^{1/2}$$

Case 2: If $\Lambda > 0$, then from (26), we have

$$\tfrac{1}{2}a^2(\beta^2+1)^{1/2}\operatorname{cosec}^2 aT > \Lambda^{1/2}$$

and (27a) is automatically satisfied if

$$\frac{9}{4(2-\beta^2)} \leq \frac{4\Lambda}{a^4}$$

From (27a), we also find that

$$\beta^2 < 2$$

The scalar of expansion θ calculated for the flow vector v^i is given by

$$\theta = \frac{2a^2\cos aT}{\sin^2 aT} \tag{28}$$

The rotation W is identically zero and the shear is given by

$$\sigma^2 = \left(\frac{a^4\cos^2 aT}{12\sin^4 aT} + \frac{\beta^2 a^4}{4\sin^4 aT}\right) \tag{29}$$

The nonvanishing components of conformal curvature tensor are

$$C^{12}_{12} = -\frac{a^4}{12\sin^2 aT}[(\beta^2+1)\operatorname{cosec}^2 aT - 6\beta\operatorname{cosec} aT\cot aT], \tag{30}$$

$$C^{13}_{13} = -\frac{a^4}{12\sin^2 aT}[(\beta^2+1)\operatorname{cosec}^2 aT + 6\beta\operatorname{cosec} aT\cot aT] \tag{31}$$

$$C^{14}_{14} = \frac{a^4}{6\sin^2 aT}[(\beta^2+1)\operatorname{cosec}^2 aT] \tag{32}$$

Hence the space-time is Petrov type D when $\beta = 0$ and nondegenerate Petrov type I otherwise.

The flow vector represents an expanding, shearing, but nonrotating universe in general.

The model (22) has an initial singularity at $T=0$. This initial singularity is point type, barrel type, or cigar type if $|\beta| \lesseqgtr 1$. It is interesting to note that the model (22) starts with big bang at $T=0$, stops at $T=\pi/2a$, and finally collapses at $T=\pi/a$. At $T=\pi/a$, the model has a singularity of cigar type.

When $T\to 0$, then

$$\sigma \to \left(\frac{1}{12}+\frac{\beta^2}{4}\right) \quad \text{and} \quad \frac{\varepsilon}{p}\to\frac{5-\beta^2}{\beta^2+1}$$

The condition $\varepsilon > p$ tends to $\beta^2 < 2$ as $T\to 0$, i.e., near the singularity. The expressions for E_4^4/ε = magnetic energy/material energy, σ/θ, ε/θ^2, and E/θ^2 are as follows:

$$\frac{E_4^4}{\varepsilon}=\frac{a^2\sin^2 aT}{(5a^2\cos^2 aT-\beta^2a^2)-(a^2-16\pi\Lambda)\sin^2 aT} \tag{33}$$

When $T\to 0$, then $E_4^4/\varepsilon\to 0$, which shows that material energy is more dominant than magnetic energy near the singularity.

$$\frac{\sigma}{\theta}=\frac{1}{2\cos aT}\left(\frac{\cos^2 aT}{12}+\frac{\beta^2}{4}\right)^{1/2} \tag{34}$$

$$\frac{\varepsilon}{\theta^2}=\frac{1}{32\pi\cos^2 aT}\left(\frac{5\cos^2 aT}{4}-\frac{\beta^2}{4}\right)-\frac{\sin^2 aT}{128\pi\cos^2 aT}+\frac{\Lambda\sin^4 aT}{32\pi a^4\cos^2 aT} \tag{35}$$

$$\frac{E}{\theta^2}=\frac{1}{16\sqrt{3}\cos^2 aT}[(\beta^2+1)^2+12\beta^2\cos^2 aT]^{1/2} \tag{36}$$

When $T\to 0$, then

$$\frac{\sigma}{\theta}\to\frac{1}{24}(3\beta^2+1)^{1/2}$$

$$\frac{\varepsilon}{\theta^2}\to\frac{1}{32\pi}\left(\frac{5}{4}-\frac{\beta^2}{4}\right)$$

and

$$\frac{E}{\theta^2}\to\frac{1}{16\sqrt{3}}[(\beta^2+1)^2+12\beta^2]^{1/2}$$

In the absence of a magnetic field, the model (23) starts with big bang at $T=0$ and continues to expand until it stops at $T=\infty$. The effect of a magnetic field is to cause the model to expand up to a finite interval of time, then collapse into a second singularity. The magnetic field also introduces inhomogeneity in pressure and density. The nonvanishing components of conformal curvature tensor in the absence of magnetic field are given by

$$C^{12}_{12} = -\frac{1}{12T^4}[(\beta^2+1)-6\beta]$$

$$C^{13}_{13} = -\frac{1}{12T^4}[(\beta^2+1)+6\beta]$$

$$C^{14}_{14} = \frac{\beta^2+1}{6T^4}$$

Thus, in the absence of a magnetic field, the space-time is nondegenerate Petrov type I and it is Petrov type D when $\beta = 0$. For large value of T, space-time is conformally flat, since $\lim_{T\to\infty}(\sigma/\theta) = 0$. Hence the model approaches isotropy for large values of T.

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